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Even and odd parts of limit periodic continued fractions

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Abstract: Even and odd parts of limit periodic continued fractions $\mathrm{K}(a_n/1)$, $a_n \to a$ are again limit periodic in most cases. For $a=\infty$ additional conditions are needed; the particular ones used in the paper imply that the even and odd parts $\mathrm{K}(c_n/1)$ and $\mathrm{K}(d_n/1)$ are such that $c_n \to -\frac{1}{4}$ and $d_n \to -\frac{1}{4}$. Illustrating examples are included.

Keywords: Limit periodic continued fractions, even and odd parts, convergence.

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1. Introduction

The present paper deals with continued fractions of the form

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1}, \quad a_n \in \mathbb{C}\setminus\{0\} \tag{1.1}$$

It is well known, that the *periodic* continued fraction of the form (1.1),

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a}{1}, \quad a \neq 0, \tag{1.2}$$

converges for all $a \in \mathbb{C}$ exept on the ray $(-\infty, -\frac{1}{4})$, where it diverges [5, Theorem 3.2]. The value of (1.2) in case of convergence is

$$x = \tfrac{1}{2}\left[\sqrt{1+4a}-1\right], \quad \operatorname{Re}\sqrt{1+4a} \geqslant 0. \tag{1.3}$$

A continued fraction (1.1), where $a_n \to a$ for $n \to \infty$, is called a *limit periodic* continued fraction. Here $a=0$ or $a=\infty$ is permitted. It is well known, that if a is not on the ray $[-\infty, -\frac{1}{4}]$ of the extended negative real axis, the limit periodic continued fraction (1.1) with $a_n \to a$ *converges* to a value in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, [6, Satz 2.41; 5, Theorem 4.45]. For $a \in (-\infty, -\frac{1}{4})$, where (1.2) diverges (by oscillation), only rather special results are known, [1,3].

We shall concentrate here on some cases where $a_n \to -\frac{1}{4}$ or $a_n \to \infty$. The main purpose is to establish a connection between these two types of limit periodic continued fractions.

The interest in these special limit periodic continued fractions is motivated by their roles in connection with the regular C-fraction expansions

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{\alpha_n z}{1}, \quad \alpha_n \neq 0, \tag{1.4}$$

of ratios of hypergeometric or confluent hypergeometric functions. For ${}_2F_1$-functions we have $\alpha_n \to -\frac{1}{4}$, whereas for ${}_2F_0$-series we have $\alpha_n \to \infty$. See for instance [5, Chapter 6] and [2, §12.5]. (We shall here not be concerned with the third case $\alpha_n \to 0$, which occurs e.g. for ${}_1F_1(b; c; z)$.)

In the case $a = -\frac{1}{4}$, much is known and much is unknown. From the parabola theorem [5, Theorem 4.42; 8; 9], it follows easily that the continued fraction converges when $a_n \to -\frac{1}{4}$, regardless of how slowly this goes, as long as all a_n from a certain n_0 on are in a $(\pi - 2\varepsilon)$ -sector with vertex at $-\frac{1}{4}$, $\varepsilon > 0$, not containing -1. Without restriction on the direction, conditions must be put on the speed. It has been known for a long time, perhaps since Pringsheim [7], that

$$|a_n + \tfrac{1}{4}| \leqslant 1/4(4n^2 - 1) \tag{1.5}$$

suffices for convergence [10, (1.3)]. In order to get a picture of how good this result is, one has been interested in what is assumed to be the worst case, i.e. when a_n approaches $-\frac{1}{4}$ from the left,

$$a_n = -\tfrac{1}{4} - \delta_n, \qquad \delta_n > 0, \quad \delta_n \to 0. \tag{1.6}$$

It is well known that the continued fraction converges if

$$\delta_n = \frac{C}{16(n+\theta)(n+\theta+1)}, \qquad \theta \notin \{-1, -2, -3, \dots\}, \tag{1.7}$$

$C \leqslant 1$, even the values of the continued fraction and its tails are known [11]. In [11] was required $\theta \geqslant -\frac{1}{2}$. The extension to any $\theta \notin \{-1, -2, -3\dots\}$ is simple. Very recently it was proved in [4], by using 'tail criteria' from [12], that (1.1) with a_n as in (1.6), (1.7), *diverges* for $C > 1$. Regular C-fractions with coefficients (1.6), (1.7) are expansions of certain hypergeometric functions [11], and they also come up in a certain procedure for solving differential equations by continued fractions [13]. For further references in the $-\frac{1}{4}$-case, see [11].

Also the case $a = \infty$ is to a certain extent taken care of by the parabola theorem. One important difference from the $-\frac{1}{4}$-case is that the speed at which $a_n \to \infty$ plays an important role, regardless of direction of approach. But also here approach along the negative real axis seems to be the worst case(among all fixed directions).

Let it finally be mentioned, that in a more general context the term periodic (limit periodic) would be called periodic (limit periodic) with perod 1, or simply 1-periodic (limit 1-periodic).

2. Even and odd part of $\mathrm{K}(a_n/1)$

From [5, Theorem 2.7] we know that any sequence $\{\varphi_n\}_{n=0}^{\infty}$ in the extended complex plane is the sequence of approximants of a continued fraction $\beta_0 + \mathrm{K}(\alpha_n/\beta_n)$ iff $\varphi_0 \neq \infty$ and $\varphi_n \neq \varphi_{n-1}$, $n = 1, 2, 3, \dots$.

Hence, if a given continued fraction $b_0 + \mathrm{K}(a_n/b_n)$ happens to have the property that the approximants satisfy

$$f_{2n} \neq f_{2n+2} \quad \text{for all } n \geqslant 0,$$

then the sequence $\{f_n^*\}_{n=0}^{\infty}$, defined by

$$f_n^* = f_{2n}, \quad n = 0, 1, 2, 3, \ldots$$

is the sequence of approximants of some continued fraction $b_0 + \mathrm{K}(a_n^*/b_n^*)$, uniquely determined up to eqiuvalence transformations. In our case, with $b_0 = 0$ and all other $b_n = 1$ (i.e. the continued fraction (1. 1)) we get from [5, formula (2.4.24)]

$$\mathrm{K}\frac{a_n^*}{b_n^*} = \frac{a_1}{1+a_2} \, \underline{\ \ } \, \frac{a_2 a_3}{1+a_3+a_4} \, \underline{\ \ } \, \frac{a_4 a_5}{1+a_5+a_6} \, \cdots . \tag{2.1}$$

If the partial denominators in (2.1) are all $\neq 0$, then (2.1) has the equivalent form

$$\mathrm{K}\frac{a_n^*}{b_n^*} \approx \mathrm{K}\frac{c_n}{1} \quad (\textit{even} \text{ part of } (1.\ 1)), \tag{2.2}$$

where

$$\begin{cases} c_1 = \dfrac{a_1}{1+a_2}, \qquad c_2 = \dfrac{-a_2 a_3}{(1+a_2)(1+a_3+a_4)}, \\ c_n = \dfrac{-a_{2n-2}a_{2n-1}}{(1+a_{2n-3}+a_{2n-2})(1+a_{2n-1}+a_{2n})} \quad \text{for } n = 3, 4, 5, \ldots . \end{cases} \tag{2.3}$$

If, on the other hand, the continued fraction is such that

$$f_{2n+1} \neq f_{2n+3} \quad \text{for all } n \geqslant 0$$

then the sequence $\{\tilde{f}_n\}$, defined by

$$\tilde{f}_n = f_{2n+1}, \quad n = 0, 1, 2, \ldots$$

is the sequence of approximants of some continued fraction $\tilde{b}_0 + \mathrm{K}(\tilde{a}_n/\tilde{b}_n)$, uniquely determined up to equivalence transformations. In our case, i.e for the continued fraction (1.1), we get from [5, formula (2.4.29) (observe the misprint)]

$$\tilde{b}_0 + \mathrm{K}\frac{\tilde{a}_n}{\tilde{b}_n} = a_1 - \frac{a_1 a_2}{1+a_2+a_3} \, \underline{\ \ } \, \frac{a_3 a_4}{1+a_4+a_5} \, \underline{\ \ } \, \frac{a_5 a_6}{1+a_6+a_7} \, - \cdots . \tag{2.4}$$

If the partial denominators in (2.4) are $\neq 0$, (2.4) has the equivalent form

$$\tilde{b}_0 + \mathrm{K}\frac{\tilde{a}_n}{\tilde{b}_n} \approx d_0 + \mathrm{K}\frac{d_n}{1} \quad (\textit{odd} \text{ part of } (1.1)), \tag{2.5}$$

where

$$\begin{cases} d_0 = a_1, \qquad d_1 = \dfrac{-a_1 a_2}{1+a_2+a_3}, \\ d_n = \dfrac{-a_{2n-1}a_{2n}}{(1+a_{2n-2}+a_{2n-1})(1+a_{2n}+a_{2n+1})} \quad \text{for } n = 2, 3, 4, \ldots . \end{cases} \tag{2.6}$$

From [5, Theorem 2.8] we know that a sequence $\{\varphi_n\}$ in the extended complex plane is the sequence of approximants of a continued fraction $b_0 + \mathrm{K}(a_n/1)$ iff $\varphi_0 \neq \infty$, $\varphi_n \neq \varphi_{n-1}$ and $\varphi_{n+1} \neq \varphi_{n-1}$, $n = 1, 2, 3, \ldots$. From this follows that $\mathrm{K}(a_n/1)$ always has an even and an odd part.

Assume that the given continued fraction (1.1) is limit periodic

$$\lim_{n\to\infty} a_n = a \neq \infty.$$

Then the even and odd parts, as given by (2.1) and (2.4), or (2.2) and (2.5), are both limit periodic. We have in particular

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} d_k = -\left(\frac{a}{1+2a}\right)^2. \tag{2.7}$$

Observe that the function

$$\omega = -\left(\frac{w}{1+2w}\right)^2 = G(w) \tag{2.8}$$

maps the ray $[-\infty, -\frac{1}{4}]$ of the negative real w-axis onto the ray $[-\infty, -\frac{1}{4}]$ of the negative real ω-axis, such that $G(-\frac{1}{4}) = G(\infty) = -\frac{1}{4}$, $G(-\frac{1}{2}) = \infty$. The complement of $[-\infty, -\frac{1}{4}]$ with respect to the w-sphere is mapped onto two copies of the complement of $[-\infty, -\frac{1}{4}]$ with respect to the ω-sphere.

If (1.1) is limit periodic with

$$\lim_{n\to\infty} a_n = \infty,$$

we can not conclude that its even or odd part, as given here, is limit periodic (counterexamples are easily constructed), except under certain additional conditions. We shall here restrict ourselves to one specific case, namely

$$\lim \frac{a_{n+1}}{a_n} = \frac{1}{\zeta}, \quad 0 \leqslant |\zeta| \leqslant 1. \tag{2.9}$$

(The case when a_{n+1}/a_n has a finite number > 1 of limit points is also easy to handle.)

It is tacitly assumed that $1 + a_2 \neq 0$, $1 + a_n + a_{n+1} \neq 0$ $(n \geqslant 2)$. Since the formulas for c_k and d_k((2.3) and (2.6)) both have the form

$$\frac{-a_n a_{n+1}}{(1+a_{n-1}+a_n)(1+a_{n+1}+a_{n+2})} = \frac{-1}{\left(\dfrac{1}{a_n}+1+\dfrac{a_{n-1}}{a_n}\right)\left(\dfrac{1}{a_{n+1}}+1+\dfrac{a_{n+2}}{a_{n+1}}\right)} \tag{2.10}$$

we find in this case that

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} d_k = \frac{-1}{(1+\zeta)\left(1+\dfrac{1}{\zeta}\right)} = \frac{-\zeta}{(1+\zeta)^2}. \tag{2.11}$$

From well known properties of the Koebe function, it follows that for any ζ in the *open* unit disk, the common value of $\lim_{k\to\infty} c_k$ and $\lim_{k\to\infty} d_k$ is in the complement of the ray $[-\infty, -\frac{1}{4}]$. Hence, by [6, Satz 2.4] or [5, Theorem 4.45], both $\mathrm{K}(c_n/1)$ and $\mathrm{K}(d_n/1)$ converge. Since $\mathrm{K}(a_n/1)$ diverges for $|\zeta| < 1$ (Stern–Stolz criterion [5, Corollary 4.20]) it follows that $\mathrm{K}(c_n/1)$ and $d_0 + \mathrm{K}(d_n/1)$ converge to *different* values in this case.

For $|\zeta| = 1$ the even and odd parts are still limit periodic, but with $\lim c_n$ and $\lim d_n$ on the ray $[-\infty, -\frac{1}{4}]$. We shall look at the case $\zeta = 1$, in which case (2.11) implies $c_n \to -\frac{1}{4}$ and $d_n \to -\frac{1}{4}$. Summarizing we have:

Proposition 2.1. *If a continued fraction* $\mathrm{K}(a_n/1)$ $a_n \neq 0$ $(n \geqslant 1)$ *is such that*

$$a_n \to \infty, \quad a_{n+1}/a_n \to 1, \quad 1 + a_2 \neq 0, \quad 1 + a_n + a_{n+1} \neq 0, \quad n \geqslant 2, \tag{2.12}$$

then its even and odd parts, $\mathrm{K}(c_n/1)$ *and* $\mathrm{K}(d_n/1)$, *are limit periodic with* $c_n \to -\frac{1}{4}$ *and* $d_n \to -\frac{1}{4}$.

Convergence results for continued fractions $\mathrm{K}(c_n/1)$ with $c_n \to -\frac{1}{4}$ may then be used to investigate convergence/divergence of the given continued fraction. Conversely, convergence statements in the '∞-case' may lead to convergence statements on '$-\frac{1}{4}$-fractions'. To illustrate the idea, we shall look at some examples.

3. A special result

Proposition 2.1 can give results on convergence/divergence of the even and odd parts of a continued fraction $\mathrm{K}(a_n/1)$, where $a_n \to \infty$ and $a_{n+1}/a_n \to 1$. We shall look at a special case of the situation in Proposition 2.1 (still assuming that $1 + a_2 \neq 0$, $1 + a_n + a_{n+1} \neq 0$, $n \geqslant 2$).

Proposition 3.1. *Let*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{1} \tag{3.1}$$

be a continued fraction with

$$a_n = \gamma_0 n^{\alpha} + \gamma_1 n^{\alpha-1} + \gamma_2 n^{\alpha-2} + o(n^{\alpha-2}), \tag{3.2}$$

where γ_0, γ_1, γ_2 *are complex,* $\gamma_0 \neq 0$, *and where* $\alpha \geqslant 2$. *Then the even and odd parts,* $\mathrm{K}(c_n/1)$ *and* $\mathrm{K}(d_n/1)$, *are such that*

$$\lim_{n\to\infty} n^2\left(c_n + \tfrac{1}{4}\right) = \lim_{n\to\infty} n^2\left(d_n + \tfrac{1}{4}\right) = \begin{cases} \dfrac{\alpha(\alpha-4)}{64} & \text{for } \alpha > 2, \\[2ex] \dfrac{\gamma_0^{-1} - 1}{16} & \text{for } \alpha = 2. \end{cases}$$

Proof. Straight forward computation of (2.3) and (2.6) with (3.2) leads to the following formulas for $\alpha > 2$:

$$c_n = -\frac{1}{4} + \frac{\alpha^2 - 4\alpha}{64} n^{-2} + o(n^{-2}),$$

$$d_n = -\frac{1}{4} + \frac{\alpha^2 - 4\alpha}{64} n^{-2} + o(n^{-2}).$$

For $\alpha = 2$ the proof is similar (even simpler). □

Remark 1. It follows immediately from Proposition 3.1 that the even and odd parts of (3.2) both converge for $\alpha > 2$. In fact

$$-\frac{1}{16} < \frac{\alpha(\alpha-4)}{64} \leqslant 0 \quad \text{for } 2 < \alpha \leqslant 4,$$

and

$$\frac{\alpha(\alpha-4)}{64}>0 \quad \text{for } \alpha>4.$$

By virtue of (1.5) we have convergence in the first case, whereas convergence in the second case follows from the fact that the sequences $\{c_n\}$ and $\{d_n\}$ both approach $-\frac{1}{4}$ from the inside of a parabola of the parabola theorem.

However, for γ_0 not a negative number, this also follows from the multiple parabola theorem [5, Theorem 4.43]. If $\alpha=2$ and γ_0 is not a negative number, the even and odd parts both converge: If $\gamma_0 \geqslant 1$ it follows from (1.5), if $\gamma_0<1$ or not real, c_n and d_n both approach $-\frac{1}{4}$ from the inside of a parabola of the parabola theorem. The case when $\alpha=2$ and γ_0 negative, such that $a_n \to \infty$ (asymptotically) along the negative real axis, is not covered by Theorem 4.43 in [5] or by Proposition 3.1.

Remark 2. It is not hard to prove, by using the parabola theorem, that (3.1) diverges for $\alpha>2$ and converges for $\alpha<2$ in (3.2) when γ_0 is not a negative number. For $\alpha=2$ the question is open. Cases of convergence are known.

We shall now study an example, where γ_0 is a negative number and $\alpha=2$. (The example is not quite of the type (3.2).)

4. An example

As an example where the idea in Proposition 2.1 leads to a conclusive conclusion on convergence we shall study the continued fraction

$$\frac{-1^2}{1}\,\underset{+}{}\,\frac{-2.4}{1}\,\underset{+}{}\,\frac{-3^2}{1}\,\underset{+}{}\,\frac{-4.6}{1}\,\underset{+}{}\cdots, \tag{4.1}$$

i.e.

$$a_{2k+1}=-(2k+1)^2, \quad k=0,1,2,\ldots,$$
$$a_{2k}=-2k(2k+2), \quad k=1,2,3,\ldots.$$

Here $a_n \in(-\infty,-1)$, such that the parabola theorem is not applicable. The even part of (4.1) is

$$\frac{\frac{1}{7}}{1}+\cfrac{-\frac{9}{28}}{1+\displaystyle\mathop{\mathrm{K}}_{k=3}^{\infty}\frac{-\frac{1}{4}-1/16k(k-1)}{1}},$$

and the odd part is

$$-1+\cfrac{\frac{1}{2}}{1+\displaystyle\mathop{\mathrm{K}}_{k=2}^{\infty}\frac{-\frac{1}{4}-1/16k(k-1)}{1}}.$$

From [11, (3,6′)] we find that both have the value 1. Hence the continued fraction (4.1) converges and has value 1.

Generally, however, Proposition 2.1 seems to be better suited for concluding *divergence* than convergence of $\mathrm{K}(a_n/1)$. If either the even or odd part diverges, so does $\mathrm{K}(a_n/1)$, and also if even and odd parts converge to different values.

The convergence of (4.1) can also be proved directly, once we see that the sequence $\{g^{(n)}\}_0^\infty$, where

$$g^{(n)} = (-1)^n \cdot (n+1),$$

is a sequence of right or wrong tails of (4.1), i.e. such that

$$g^{(n)} = \frac{a_{n+1}}{1+g^{(n+1)}}, \quad n = 0, 1, 2, \ldots .$$

Then it follows from [12, Theorem 1] that (4.1) converges to $g^{(0)}$, and has the (right) tails $g^{(n)}$.

The example studied in the present section is an extremely slowly converging continued fraction. By using the values of the tails and results from [12] we can find pretty good estimates for the truncation error and thus for f_n.

With

$$\kappa_n = -\frac{1+g^{(n)}}{g^{(n)}} = -1 + \frac{(-1)^{n+1}}{n+1}$$

we find (since $\kappa_{2p}\kappa_{2p+1} = 1$ and $1+\kappa_{2p} = -1/(2p+1)$)

$$1+\kappa_1+\kappa_1\kappa_2+\cdots+\kappa_1\kappa_2\cdots\kappa_{2k} = 1 + \frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k+1}\right),$$

$$1+\kappa_1+\kappa_1\kappa_2+\cdots+\kappa_1\kappa_2\cdots\kappa_{2k+1} = \frac{1}{2}\left(1+\frac{1}{3}+\cdots+\frac{1}{2k+1}\right). \tag{4.3}$$

From the formula (1.8) in [12] we have, when f_n means the nth approximant:

$$f_n - g^{(0)} = \frac{-g^{(0)}}{1+\kappa_1+\kappa_1\kappa_2+\cdots+\kappa_1\kappa_2\cdots\kappa_n}. \tag{4.4}$$

From this and (4.3) follows, that (4.1) converges to 1, actually that $1, -2, 3, -4, \ldots$ is the sequence of *right* tails for the continued fraction, beginning with the value itself. (4.3) and (4.4) even give a rather good estimate for the truncation error. We have for instance, since

$$\frac{1}{2}\left(1+\frac{1}{2}\log\frac{2k+3}{3}\right) < \frac{1}{2}\left(1+\frac{1}{3}+\cdots+\frac{1}{2k+1}\right) < \frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{2}\log\frac{2k+1}{3}\right),$$

and in particular for $k = 10^5$

$$3.2768 < \frac{1}{2}\left(1+\frac{1}{3}+\cdots+\frac{1}{200001}\right) < 3.4436,$$

that

$$0.2903 < 1 - f_{200001} < 0{,}3052$$

and hence

$$0.6948 < f_{200001} < 0.7097.$$

Numerical computation, single precision, on a VAX 11/750 computer, using the backwards recurrence algorithm (BRA) gives

$$f_{200001} = 0.703186.$$

5. Final remarks

The present paper merely indicates, essentially through examples, how even and odd parts are potential tools in the discussion of convergence/divergence of limit periodic continued fractions $\mathrm{K}(a_n/1)$ where $a_n \to \infty$ or $a_n \to -\frac{1}{4}$. But there are many open questions: If one tries to apply the method of even and odd parts to continued fractions like for instance $\mathrm{K}_{n=1}^{\infty}(-n^2/1)$, one is led to limit-periodic continued fractions ($c_n \to -\frac{1}{4}$, $d_n \to -\frac{1}{4}$), but where no established result on $-\frac{1}{4}$-fractions exists. For the parameters of the even and odd parts we get

$$\begin{aligned} c_k &= -\tfrac{1}{4} - \frac{1}{8k(k-\frac{3}{2})}, \quad k \geqslant 3, \\ d_k &= -\tfrac{1}{4} - \frac{1}{8(k-1)(k+\frac{1}{2})}, \quad k \geqslant 2. \end{aligned} \tag{5.1}$$

A nearby (and perhaps good) guess is that even and odd parts both diverge, since we have a smaller factor that 16 in the denominator. But since neither of the formulas (5.1) is of form (1.6), (1.7), the result in [4] is not applicable. (There is strong evidence, numerical and other, that $\mathrm{K}_{n=1}^{\infty}(-n^2/1)$ diverges.)

This shows the need for an extension of the result in [4] to other types. But this is likely to be difficult. The proof in [4] was highly dependent upon explicit knowledge of a sequence of tails (right or wrong).

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References

(1) J. Gill, Infinite compositions of Möbius transformations, *Trans. Amer. Math. Soc.* **176** (1973) 479–487.
(2) P. Henrici, *Applied and Computational Complex Analysis, Vol. 2* (Wiley, New York, 1977).
(3) L. Jacobsen, Convergence of limit k-periodic continued fractions $\mathrm{K}(a_n/b_n)$ and of subsequences of their tails, *Proc. London Math. Soc.* **51**(3) (1985) 563–576.
(4) L. Jacobsen and A. Magnus, On the convergence of limit periodic continued fractions $\mathrm{K}(a_n/1)$, where $a_n \to -\frac{1}{4}$, in: P.R. Graves-Morris, E.B. Saff and R.S. Varga, Eds., *Rational Approximation And Interpolation,* Proceedings Tampa, FL,1983, Lecture Notes in Mathematics **1105** (Springer, New York, 1984).
(5) W.B. Jones and W.J. Thron, *Continued Fractions, Analytic Theory and Applications,* Encyclopedia of Mathematics **11** (Addison-Wesley, Reading, MA, 1980).
(6) O. Perron, *Die Lehre von den Kettenbrüchen, 3. Aufl., 2. Band* (Stuttgart, Teubner, 1957).
(7) A. Pringsheim, Über die Konvergenz unendlicher Kettenbrüche, *Sitzungsberichte d. Bayer. Akad. Wiss. Math.-Nat. Klasse* **28** (1899) 295-324.
(8) W.J. Thron, On parabolic convergence regions for continued fractions *Math. Z.* **69**, 173–182.
(9) W.J. Thron, Convergence of sequences of linear fractional transformations and of continued fractions, *J. Indian Math. 27* (1963) 103–127.
(10) W.J. Thron and H. Waadeland, Accelerating convergence of limit periodic continued fractions $\mathrm{K}(a_n/1)$, *Numer.Math.* **34** (1980) 155–170.

(11) W.J. Thron and H. Waadeland, On a certain transformation of continued fractions, in: W.B. Jones, W.J. Thron and H. Waadeland, Eds., *Analytic Theory of Continued Fractions*, Proceedings Loen, Norway 1981, Lecture Notes in Mathematics **932** (Springer, New York, 1982).
(12) Haakon Waadeland, Tales about tails, *Proc. Amer. Math. Soc.* **90** (January 1984)
(13) H. Waadeland, Differential equations and modifications of continued fractions, some simple observations, in: H. Waadeland and H. Wallin, *Padé Approximants and Continued Fractions*, Proceedings Undersaker, Sweden, 1982, Det. Kgl. Norske Vid. Selsk. Skr. **1** (Universitetsforlaget, Trondheim, 1983).